8.6 Integration by Tables and Other Integration Techniques

- **Evaluate an indefinite integral using a table of integrals.**
- **Evaluate an indefinite integral using reduction formulas.**
- **Evaluate an indefinite integral involving rational functions of sine and cosine.**

Integration by Tables

So far in this chapter, you have studied several integration techniques that can be used with the basic integration rules. But merely knowing *how* to use the various techniques is not enough. You also need to know *when* to use them. Integration is first and foremost a problem of recognition. That is, you must recognize which rule or technique to apply to obtain an antiderivative. Frequently, a slight alteration of an integrand will require a different integration technique (or produce a function whose antiderivative is not an elementary function), as shown below.

$$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C \qquad \text{Integration by parts}$$

$$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C \qquad \text{Power Rule}$$

$$\int \frac{1}{x \ln x}\,dx = \ln|\ln x| + C \qquad \text{Log Rule}$$

$$\int \frac{x}{\ln x}\,dx = ? \qquad \text{Not an elementary function}$$

▷ **TECHNOLOGY** A computer algebra system consists, in part, of a database of integration formulas. The primary difference between using a computer algebra system and using tables of integrals is that with a computer algebra system, the computer searches through the database to find a fit. With integration tables, *you* must do the searching.

Many people find tables of integrals to be a valuable supplement to the integration techniques discussed in this chapter. Tables of common integrals can be found in Appendix B. **Integration by tables** is not a "cure-all" for all of the difficulties that can accompany integration—using tables of integrals requires considerable thought and insight and often involves substitution.

Each integration formula in Appendix B can be developed using one or more of the techniques in this chapter. You should try to verify several of the formulas. For instance, Formula 4

$$\int \frac{u}{(a + bu)^2}\,du = \frac{1}{b^2}\left(\frac{a}{a + bu} + \ln|a + bu|\right) + C \qquad \text{Formula 4}$$

can be verified using the method of partial fractions, Formula 19

$$\int \frac{\sqrt{a + bu}}{u}\,du = 2\sqrt{a + bu} + a\int \frac{du}{u\sqrt{a + bu}} \qquad \text{Formula 19}$$

can be verified using integration by parts, and Formula 84

$$\int \frac{1}{1 + e^u}\,du = u - \ln(1 + e^u) + C \qquad \text{Formula 84}$$

can be verified using substitution. Note that the integrals in Appendix B are classified according to the form of the integrand. Several of the forms are listed below.

u^n	$(a + bu)$
$(a + bu + cu^2)$	$\sqrt{a + bu}$
$(a^2 \pm u^2)$	$\sqrt{u^2 \pm a^2}$
$\sqrt{a^2 - u^2}$	Trigonometric functions
Inverse trigonometric functions	Exponential functions
Logarithmic functions	

Exploration

Use the tables of integrals in Appendix B and the substitution

$$u = \sqrt{x-1}$$

to evaluate the integral in Example 1. When you do this, you should obtain

$$\int \frac{dx}{x\sqrt{x-1}} = \int \frac{2\,du}{u^2+1}.$$

Does this produce the same result as that obtained in Example 1?

EXAMPLE 1 Integration by Tables

Find $\displaystyle\int \frac{dx}{x\sqrt{x-1}}$.

Solution Because the expression inside the radical is linear, you should consider forms involving $\sqrt{a+bu}$.

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \arctan \sqrt{\frac{a+bu}{-a}} + C \qquad \text{Formula 17 } (a < 0)$$

Let $a = -1$, $b = 1$, and $u = x$. Then $du = dx$, and you can write

$$\int \frac{dx}{x\sqrt{x-1}} = 2 \arctan \sqrt{x-1} + C.$$

EXAMPLE 2 Integration by Tables

See LarsonCalculus.com for an interactive version of this type of example.

Find $\displaystyle\int x\sqrt{x^4-9}\,dx$.

Solution Because the radical has the form $\sqrt{u^2-a^2}$, you should consider Formula 26.

$$\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2 \ln\left|u + \sqrt{u^2-a^2}\right|\right) + C$$

Let $u = x^2$ and $a = 3$. Then $du = 2x\,dx$, and you have

$$\int x\sqrt{x^4-9}\,dx = \frac{1}{2}\int \sqrt{(x^2)^2-3^2}\,(2x)\,dx$$

$$= \frac{1}{4}\left(x^2\sqrt{x^4-9} - 9\ln\left|x^2+\sqrt{x^4-9}\right|\right) + C.$$

EXAMPLE 3 Integration by Tables

Evaluate $\displaystyle\int_0^2 \frac{x}{1+e^{-x^2}}\,dx$.

Solution Of the forms involving e^u, consider the formula

$$\int \frac{du}{1+e^u} = u - \ln(1+e^u) + C. \qquad \text{Formula 84}$$

Let $u = -x^2$. Then $du = -2x\,dx$, and you have

$$\int \frac{x}{1+e^{-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x\,dx}{1+e^{-x^2}}$$

$$= -\frac{1}{2}\left[-x^2 - \ln\left(1+e^{-x^2}\right)\right] + C$$

$$= \frac{1}{2}\left[x^2 + \ln\left(1+e^{-x^2}\right)\right] + C.$$

So, the value of the definite integral is

$$\int_0^2 \frac{x}{1+e^{-x^2}}\,dx = \frac{1}{2}\left[x^2 + \ln\left(1+e^{-x^2}\right)\right]_0^2 = \frac{1}{2}\left[4 + \ln\left(1+e^{-4}\right) - \ln 2\right] \approx 1.66.$$

Figure 8.14 shows the region whose area is represented by this integral.

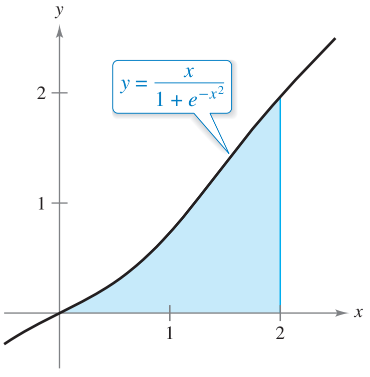


Figure 8.14

Reduction Formulas

Several of the integrals in the integration tables have the form

$$\int f(x)\,dx = g(x) + \int h(x)\,dx.$$

Such integration formulas are called **reduction formulas** because they reduce a given integral to the sum of a function and a simpler integral.

EXAMPLE 4 Using a Reduction Formula

Find $\int x^3 \sin x\,dx$.

Solution Consider the three formulas listed below.

$$\int u \sin u\,du = \sin u - u\cos u + C \qquad \text{Formula 52}$$

$$\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du \qquad \text{Formula 54}$$

$$\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du \qquad \text{Formula 55}$$

Using Formula 54, Formula 55, and then Formula 52 produces

$$\begin{aligned}\int x^3 \sin x\,dx &= -x^3\cos x + 3\int x^2\cos x\,dx\\ &= -x^3\cos x + 3\left(x^2\sin x - 2\int x\sin x\,dx\right)\\ &= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.\end{aligned}$$

EXAMPLE 5 Using a Reduction Formula

Find $\displaystyle\int \frac{\sqrt{3-5x}}{2x}\,dx$.

Solution Consider the two formulas listed below.

$$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C \qquad \text{Formula 17 } (a > 0)$$

$$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}} \qquad \text{Formula 19}$$

Using Formula 19, with $a = 3$, $b = -5$, and $u = x$, produces

$$\begin{aligned}\frac{1}{2}\int \frac{\sqrt{3-5x}}{x}\,dx &= \frac{1}{2}\left(2\sqrt{3-5x} + 3\int \frac{dx}{x\sqrt{3-5x}}\right)\\ &= \sqrt{3-5x} + \frac{3}{2}\int \frac{dx}{x\sqrt{3-5x}}.\end{aligned}$$

Using Formula 17, with $a = 3$, $b = -5$, and $u = x$, produces

$$\begin{aligned}\int \frac{\sqrt{3-5x}}{2x}\,dx &= \sqrt{3-5x} + \frac{3}{2}\left(\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3-5x}-\sqrt{3}}{\sqrt{3-5x}+\sqrt{3}}\right|\right) + C\\ &= \sqrt{3-5x} + \frac{\sqrt{3}}{2}\ln\left|\frac{\sqrt{3-5x}-\sqrt{3}}{\sqrt{3-5x}+\sqrt{3}}\right| + C.\end{aligned}$$

TECHNOLOGY Sometimes when you use computer algebra systems, you obtain results that look very different, but are actually equivalent. Here is how two different systems evaluated the integral in Example 5.

Maple

$\sqrt{3-5x} -$
$\sqrt{3}\,\text{arctanh}\left(\frac{1}{3}\sqrt{3-5x}\sqrt{3}\right)$

Mathematica

$\sqrt{3-5x} -$
$\sqrt{3}\,\text{ArcTanh}\left[\sqrt{1-\frac{5x}{3}}\right]$

Notice that computer algebra systems do not include a constant of integration.

Rational Functions of Sine and Cosine

EXAMPLE 6 **Integration by Tables**

Find $\int \frac{\sin 2x}{2 + \cos x}\,dx.$

Solution Substituting $2 \sin x \cos x$ for $\sin 2x$ produces

$$\int \frac{\sin 2x}{2 + \cos x}\,dx = 2\int \frac{\sin x \cos x}{2 + \cos x}\,dx.$$

A check of the forms involving $\sin u$ or $\cos u$ in Appendix B shows that none of those listed applies. So, you can consider forms involving $a + bu$. For example,

$$\int \frac{u\,du}{a + bu} = \frac{1}{b^2}(bu - a\ln|a + bu|) + C.$$ Formula 3

Let $a = 2$, $b = 1$, and $u = \cos x$. Then $du = -\sin x\,dx$, and you have

$$\begin{aligned}2\int \frac{\sin x \cos x}{2 + \cos x}\,dx &= -2\int \frac{\cos x(-\sin x\,dx)}{2 + \cos x}\\ &= -2(\cos x - 2\ln|2 + \cos x|) + C\\ &= -2\cos x + 4\ln|2 + \cos x| + C.\end{aligned}$$

Example 6 involves a rational expression of $\sin x$ and $\cos x$. When you are unable to find an integral of this form in the integration tables, try using the following special substitution to convert the trigonometric expression to a standard rational expression.

Substitution for Rational Functions of Sine and Cosine

For integrals involving rational functions of sine and cosine, the substitution

$$u = \frac{\sin x}{1 + \cos x} = \tan\frac{x}{2}$$

yields

$$\cos x = \frac{1 - u^2}{1 + u^2}, \quad \sin x = \frac{2u}{1 + u^2}, \quad \text{and} \quad dx = \frac{2\,du}{1 + u^2}.$$

Proof From the substitution for u, it follows that

$$u^2 = \frac{\sin^2 x}{(1 + \cos x)^2} = \frac{1 - \cos^2 x}{(1 + \cos x)^2} = \frac{1 - \cos x}{1 + \cos x}.$$

Solving for $\cos x$ produces $\cos x = (1 - u^2)/(1 + u^2)$. To find $\sin x$, write $u = \sin x/(1 + \cos x)$ as

$$\sin x = u(1 + \cos x) = u\left(1 + \frac{1 - u^2}{1 + u^2}\right) = \frac{2u}{1 + u^2}.$$

Finally, to find dx, consider $u = \tan(x/2)$. Then you have $\arctan u = x/2$ and

$$dx = \frac{2\,du}{1 + u^2}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

8.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Integration by Tables In Exercises 1 and 2, use a table of integrals with forms involving $a + bu$ to find the indefinite integral.

1. $\displaystyle\int \frac{x^2}{5 + x}\,dx$

2. $\displaystyle\int \frac{2}{x^2(4 + 3x)^2}\,dx$

Integration by Tables In Exercises 3 and 4, use a table of integrals with forms involving $\sqrt{a^2 - u^2}$ to find the indefinite integral.

3. $\displaystyle\int \frac{1}{x^2\sqrt{1 - x^2}}\,dx$

4. $\displaystyle\int \frac{\sqrt{64 - x^4}}{x}\,dx$

Integration by Tables In Exercises 5–8, use a table of integrals with forms involving the trigonometric functions to find the indefinite integral.

5. $\displaystyle\int \cos^4 3x\,dx$

6. $\displaystyle\int \frac{\sin^4\sqrt{x}}{\sqrt{x}}\,dx$

7. $\displaystyle\int \frac{1}{\sqrt{x}\left(1 - \cos\sqrt{x}\right)}\,dx$

8. $\displaystyle\int \frac{1}{1 + \cot 4x}\,dx$

Integration by Tables In Exercises 9 and 10, use a table of integrals with forms involving e^u to find the indefinite integral.

9. $\displaystyle\int \frac{1}{1 + e^{2x}}\,dx$

10. $\displaystyle\int e^{-4x}\sin 3x\,dx$

Integration by Tables In Exercises 11 and 12, use a table of integrals with forms involving $\ln u$ to find the indefinite integral.

11. $\displaystyle\int x^7 \ln x\,dx$

12. $\displaystyle\int (\ln x)^3\,dx$

Using Two Methods In Exercises 13–16, find the indefinite integral (a) using integration tables and (b) using the given method.

Integral	Method
13. $\displaystyle\int x^2e^{3x}\,dx$	Integration by parts
14. $\displaystyle\int x^5 \ln x\,dx$	Integration by parts
15. $\displaystyle\int \frac{1}{x^2(x + 1)}\,dx$	Partial fractions
16. $\displaystyle\int \frac{1}{x^2 - 36}\,dx$	Partial fractions

Finding an Indefinite Integral In Exercises 17–38, use integration tables to find the indefinite integral.

17. $\displaystyle\int x\,\text{arccsc}(x^2 + 1)\,dx$

18. $\displaystyle\int \arcsin 4x\,dx$

19. $\displaystyle\int \frac{1}{x^2\sqrt{x^2 - 4}}\,dx$

20. $\displaystyle\int \frac{1}{x^2 + 4x + 8}\,dx$

21. $\displaystyle\int \frac{4x}{(2 - 5x)^2}\,dx$

22. $\displaystyle\int \frac{\theta^3}{1 + \sin\theta^4}\,d\theta$

23. $\displaystyle\int e^x \arccos e^x\,dx$

24. $\displaystyle\int \frac{e^x}{1 - \tan e^x}\,dx$

25. $\displaystyle\int \frac{x}{1 - \sec x^2}\,dx$

26. $\displaystyle\int \frac{1}{t[1 + (\ln t)^2]}\,dt$

27. $\displaystyle\int \frac{\cos\theta}{3 + 2\sin\theta + \sin^2\theta}\,d\theta$

28. $\displaystyle\int x^2\sqrt{2 + 9x^2}\,dx$

29. $\displaystyle\int \frac{1}{x^2\sqrt{2 + 9x^2}}\,dx$

30. $\displaystyle\int \sqrt{x}\arctan x^{3/2}\,dx$

31. $\displaystyle\int \frac{\ln x}{x(3 + 2\ln x)}\,dx$

32. $\displaystyle\int \frac{e^x}{(1 - e^{2x})^{3/2}}\,dx$

33. $\displaystyle\int \frac{x}{(x^2 - 6x + 10)^2}\,dx$

34. $\displaystyle\int \sqrt{\frac{5 - x}{5 + x}}\,dx$

35. $\displaystyle\int \frac{x}{\sqrt{x^4 - 6x^2 + 5}}\,dx$

36. $\displaystyle\int \frac{\cos x}{\sqrt{\sin^2 x + 1}}\,dx$

37. $\displaystyle\int \frac{e^{3x}}{(1 + e^x)^3}\,dx$

38. $\displaystyle\int \cot^4\theta\,d\theta$

Evaluating a Definite Integral In Exercises 39–46, use integration tables to evaluate the definite integral.

39. $\displaystyle\int_0^1 xe^{x^2}\,dx$

40. $\displaystyle\int_0^4 \frac{x}{\sqrt{3 + 2x}}\,dx$

41. $\displaystyle\int_1^2 x^4\ln x\,dx$

42. $\displaystyle\int_0^{\pi/2} x\sin 2x\,dx$

43. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx$

44. $\displaystyle\int_0^5 \frac{x^2}{(5 + 2x)^2}\,dx$

45. $\displaystyle\int_0^{\pi/2} t^3\cos t\,dt$

46. $\displaystyle\int_0^3 \sqrt{x^2 + 16}\,dx$

Verifying a Formula In Exercises 47–52, verify the integration formula.

47. $\displaystyle\int \frac{u^2}{(a + bu)^2}\,du = \frac{1}{b^3}\left(bu - \frac{a^2}{a + bu} - 2a\ln|a + bu|\right) + C$

48. $\displaystyle\int \frac{u^n}{\sqrt{a + bu}}\,du = \frac{2}{(2n + 1)b}\left(u^n\sqrt{a + bu} - na\int \frac{u^{n-1}}{\sqrt{a + bu}}\,du\right)$

49. $\displaystyle\int \frac{1}{(u^2 \pm a^2)^{3/2}}\,du = \frac{\pm u}{a^2\sqrt{u^2 \pm a^2}} + C$

50. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

51. $\displaystyle\int \arctan u\,du = u\arctan u - \ln\sqrt{1 + u^2} + C$

52. $\displaystyle\int (\ln u)^n\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$

Finding or Evaluating an Integral In Exercises 53–60, find or evaluate the integral.

53. $\displaystyle\int \frac{1}{2 - 3\sin\theta}\,d\theta$

54. $\displaystyle\int \frac{\sin\theta}{1 + \cos^2\theta}\,d\theta$

55. $\displaystyle\int_0^{\pi/2} \frac{1}{1 + \sin\theta + \cos\theta}\,d\theta$

56. $\displaystyle\int_0^{\pi/2} \frac{1}{3 - 2\cos\theta}\,d\theta$

57. $\displaystyle\int \frac{\sin\theta}{3 - 2\cos\theta}\,d\theta$

58. $\displaystyle\int \frac{\cos\theta}{1 + \cos\theta}\,d\theta$

59. $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

60. $\displaystyle\int \frac{4}{\csc\theta - \cot\theta}\,d\theta$

Area In Exercises 61 and 62, find the area of the region bounded by the graphs of the equations.

61. $y = \dfrac{x}{\sqrt{x+3}},\ y = 0,\ x = 6$

62. $y = \dfrac{x}{1 + e^{x^2}},\ y = 0,\ x = 2$

WRITING ABOUT CONCEPTS

63. Finding a Pattern

(a) Evaluate $\int x^n \ln x\,dx$ for $n = 1, 2,$ and 3. Describe any patterns you notice.

(b) Write a general rule for evaluating the integral in part (a), for an integer $n \geq 1$.

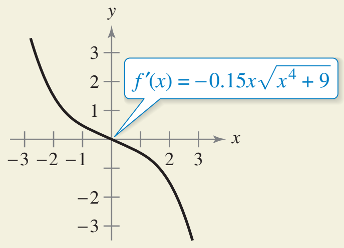
64. Reduction Formula Describe what is meant by a reduction formula. Give an example.

65. Choosing a Method State (if possible) the method or integration formula you would use to find the antiderivative. Explain why you chose that method or formula. Do not integrate.

(a) $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$ (b) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$ (c) $\displaystyle\int xe^{x^2}\,dx$

(d) $\displaystyle\int xe^x\,dx$ (e) $\displaystyle\int e^{x^2}\,dx$ (f) $\displaystyle\int e^{2x}\sqrt{e^{2x}+1}\,dx$

66. **HOW DO YOU SEE IT?** Use the graph of f' shown in the figure to answer the following.



(a) Approximate the slope of f at $x = -1$. Explain.

(b) Approximate any open intervals in which the graph of f is increasing and any open intervals in which it is decreasing. Explain.

True or False? In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

67. To use a table of integrals, the integral you are evaluating must appear in the table.

68. When using a table of integrals, you may have to make substitutions to rewrite your integral in the form in which it appears in the table.

69. Work A hydraulic cylinder on an industrial machine pushes a steel block a distance of x feet $(0 \leq x \leq 5)$, where the variable force required is $F(x) = 2000xe^{-x}$ pounds. Find the work done in pushing the block the full 5 feet through the machine.

70. Work Repeat Exercise 69, using $F(x) = \dfrac{500x}{\sqrt{26 - x^2}}$ pounds.

71. Volume Consider the region bounded by the graphs of

$$y = x\sqrt{16 - x^2},\quad y = 0,\quad x = 0,\quad \text{and}\quad x = 4.$$

Find the volume of the solid generated by revolving the region about the y-axis.

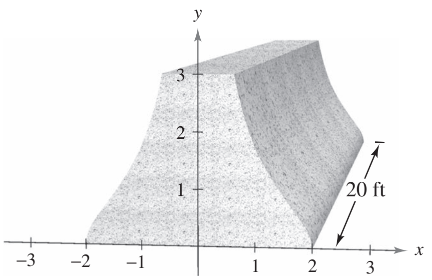
72. Building Design The cross section of a precast concrete beam for a building is bounded by the graphs of the equations

$$x = \frac{2}{\sqrt{1+y^2}},\quad x = \frac{-2}{\sqrt{1+y^2}},\quad y = 0,\quad \text{and}\quad y = 3$$

where x and y are measured in feet. The length of the beam is 20 feet (see figure).

(a) Find the volume V and the weight W of the beam. Assume the concrete weighs 148 pounds per cubic foot.

(b) Find the centroid of a cross section of the beam.



73. Population A population is growing according to the logistic model

$$N = \frac{5000}{1 + e^{4.8 - 1.9t}}$$

where t is the time in days. Find the average population over the interval $[0, 2]$.

PUTNAM EXAM CHALLENGE

74. Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$.